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LIGHT ENERGY REACHING THE
GRATING OF A SPECTROMETER

W. E. Bair

ARO, Inc.

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VON KÁRMÁN GAS DYNAMICS FACILITY
ARNOLD ENGINEERING DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
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NOTICES

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GRATING OF A SPECTROMETER

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FOREWORD

The work reported herein was sponsored by Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), Arnold Air Force Station, Tennessee under Program Element 62405334, Project 8952.

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This technical report has been reviewed and is approved.

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ABSTRACT

The effective field of view of a spectrometer, when used with an external collecting lens, is shown to be defined by the width and height of the image of the slit. The energy, E_λ, of wavelength λ, which reaches the grating of the spectrometer, is very closely approximated as

$$E_\lambda = \frac{wh}{\pi} \left[\sin^{-1} \left(\frac{d^2}{4\bar{z}^2 + d^2} \right) \right] I_\lambda \, dz$$

where

w = width of slit image

h = height of slit image

d = width of lens stop

$\bar{z}$ = distance from slit image to lens

I_λ = intensity of light emitted

z = coordinate distance from slit image toward lens

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NOMENCLATURE

d	Width of lens stop
d_c	Width of collimator image
d_c'	Width of collimator
dv	Element of volume
E_λ	Energy of wavelength λ reaching grating, per unit time

f	Focal length of lens
h	Height of slit image
h'	Height of slit
I_λ	Intensity of light emitted, w/cm^3
w	Width of slit image
w'	Width of slit
x	Horizontal distance from optical axis
y	Vertical distance from optical axis
z	Distance from slit image toward lens
$\bar{z}$	Distance from slit image to lens
z'	Distance from slit to lens
z_c	Distance from slit image to collimator image
z_c'	Distance from slit to collimator
α	Transformed horizontal variable
β	Transformed vertical variable
γ	Angle from line parallel to axis to right stop
δ	Angle from line parallel to axis to left stop
λ	Wavelength of light observed
ϕ	Angle from line parallel to axis to upper stop
ψ	Angle from line parallel to axis to lower stop
Ω	Solid angle

SECTION I
PRIMARY CONSIDERATIONS

In order to utilize the maximum resolution capability of a grating spectrometer, it is necessary to illuminate the entire ruled area of the grating. The collimating mirrors of a well-designed spectrometer are sized to fully illuminate the grating. To prevent stray light in the spectrometer from causing false indications, it is desirable to select a lens stop and slit-to-lens distance which will cause full illumination of the collimator without overillumination. When this is accomplished, the solid angle of the lens stop will be the same as the solid angle of the collimator when both are viewed from the slit. The solid angle of the lens stop will then be the same as the solid angle of the collimator image when viewed from the slit image. This is the condition chosen for the development which follows.

SECTION II
DEVELOPMENT

The energy which reaches the grating of the spectrometer is determined by integrating, over the field of emission, the intensity of the emitted radiation which reaches the grating.

$$E_\lambda = \int_v \frac{1}{4\pi} \ I_\lambda \ \Omega \ dv \tag{1}$$

The effective solid angle through which the radiation will reach the grating is limited by (1) the slit image, (2) the collimator image, or (3) the collecting lens stop. In the following analysis, the collecting lens stop and the collimator image are both assumed to be square with widths d and d_c, respectively. For this case the effective solid angle will be that subtended by a rectangle.

As shown in Fig. 1, the field of emission is divided by planes into regions for which various combinations control the solid angle. Point A, for example, is controlled on its upper angle, ϕ, by the lens and on its lower angle, ψ, by the collimator image, whereas the horizontal angles, γ and δ, are controlled by the slit image on the right and the lens on the left.

The effective rectangle for point A is shown projected on the slit image, collimator image, and lens planes in Fig. 2. In terms of the

limiting angles ϕ, ψ, γ, and δ, the solid angle subtended by the rectangle is

$$\Omega = \sin^{-1}(\sin\phi \sin\gamma) + \sin^{-1}(\sin\phi \sin\delta)$$

$$+ \sin^{-1}(\sin\psi \sin\gamma) + \sin^{-1}(\sin\psi \sin\delta) \tag{2}$$

In general, it is sufficiently accurate to assume that

$$\partial^2 I_\lambda / \partial x^2 = 0 \quad \text{and} \quad \partial^2 I_\lambda / \partial y^2 = 0 \tag{3}$$

By assuming that the second derivatives are zero and that variations in I_y transverse to the optical axis are linear in x and y, each element of volume (dx dy dz) off of the axis may be summed with the other three elements whose x and y coordinates have the same magnitudes. The total contribution of these four elements, which by the symmetry of position each emits to the grating through the same solid angle, will be the same as though each possessed an intensity equal to the mean intensity for the z-plane in which they reside. This results in the symmetry which permits treatment of I_λ as a function of z alone and also permits the evaluation of the total solid angle as

$$\Omega = 4\sin^{-1}(\sin\gamma \sin\phi) \tag{4}$$

Substituting

$$\sin\gamma = \tan\gamma / \sqrt{1 + \tan^2\gamma} \quad \text{and} \quad \sin\phi = \tan\phi / \sqrt{1 + \tan^2\phi}$$

the integral becomes

$$E_\lambda = \int \frac{I_\lambda}{\pi} \iint \sin^{-1} \frac{\tan\gamma \, \tan\phi}{\sqrt{(1 + \tan^2\gamma)(1 + \tan^2\phi)}} \, dy \, dx \, dz \tag{5}$$

Table I shows the equations of the limiting planes, x_1, x_2..., and Table II shows the values of γ and ϕ for each range of z and x or y as appropriate. The values for $z > z_c$ are not included since this is assumed to be out of the field of emission, as shown in Fig. 3. (In order to experimentally apply the derived result of this report, the location of the collimator image should be determined.)

Upon substitution of the appropriate values of $\tan\gamma$ and $\tan\phi$, along with the accompanying limits, Eq. (4) expands to nine double integrals for each range of z. The first of these double integrals, for $z < 0$, is

$$\int_{x_4}^{x_3} \int_{y_4}^{y_3} \sin^{-1} \frac{\tan\gamma_{43} \, \tan\phi_{43}}{\sqrt{(1 + \tan^2\gamma_{43})(1 + \tan^2\phi_{43})}} \, dy \, dx$$

This and the other integrals can be simplified by substituting α = tan γ and β = tan ϕ. The first integral then becomes

$$(\bar{z} - z)^2 \int_{\frac{w+d}{2\bar{z}}}^{\frac{d}{2\bar{z}}} \int_{\frac{h+d}{2\bar{z}}}^{\frac{d}{2\bar{z}}} \sin^{-1} \frac{\alpha \beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}} \, d\beta \, d\alpha \qquad (6)$$

Similar substitution in the other integrals will result in generation of various functions of z for the various limits. The integrands will be identical. Table III lists these functions and the corresponding limits.

Since the arcsin is an "odd" function, integration from

$$- \frac{d+w}{2\bar{z}} \quad \text{to} \quad \frac{d+w}{2\bar{z}} \quad \text{or} \quad \text{from} \quad - \frac{d_c+w}{2z_c} \quad \text{to} \quad \frac{d_c+w}{2z_c}$$

will yield zero. After eliminating these zero terms, the functions of z become constants as shown in Table IV. Inasmuch as the constants are identical for both ranges of z, the complete integral can be written as

$$E_\lambda = \int \frac{I_\lambda}{\pi} \left[z_c^2 \int_{\frac{d_c-w}{2z_c}}^{\frac{d}{2\bar{z}}} \int_{\frac{d_c-h}{2z_c}}^{\frac{d}{2\bar{z}}} \sin^{-1} \frac{\alpha \beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}} \, d\beta \, d\alpha + z_c \bar{z} \int_{\frac{d_c-w}{2z_c}}^{\frac{d}{2\bar{z}}} \int_{\frac{d}{2\bar{z}}}^{\frac{d+h}{2\bar{z}}} \sin^{-1} \right.$$

$$\left. \cdots d\beta \, d\alpha + z_c \bar{z} + \int_{\frac{d}{2\bar{z}}}^{\frac{d+w}{2\bar{z}}} \int_{\frac{d_c-h}{2z_c}}^{\frac{d}{2\bar{z}}} \sin^{-1} \cdots d\beta \, d\alpha + \bar{z}^2 \int_{\frac{d}{2\bar{z}}}^{\frac{d+w}{2\bar{z}}} \int_{\frac{d}{2\bar{z}}}^{\frac{d+h}{2\bar{z}}} \sin^{-1} \cdots d\beta \, d\alpha \right] dz \qquad (7)$$

These four remaining integrals can be evaluated by integration by parts setting

$$u = \sin^{-1} \frac{\alpha \beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}} \quad \text{and} \quad dv = d\beta$$

in the first integration and similarly in the second. It is of interest to note, however, that the range of integration in each integral is small and that a close approximation can be obtained by fixing the integrand at the value corresponding to the center of the overall range, $\alpha = \beta = \frac{d}{2\bar{z}}$. Upon evaluation in this manner, Eq. (1) finally becomes

$$E_\lambda = \frac{wh}{\pi} \left[\sin^{-1} \left(\frac{d^2}{4\bar{z}^2 + d^2} \right) \right] \int_{-z}^{z} I_\lambda \, dz \qquad (8)$$

Equation (8) may be expressed in terms of the actual slit dimensions, w' and h', the distance from the slit to the lens, z', and the focal length of the lens, f. In these terms, it becomes

$$E_\lambda = \frac{w'h'}{\pi} \left(\frac{f}{z'-f}\right)^2 \left[\sin^{-1}\left(\frac{(z'-f)^2 d^2}{4f^2 z'^2 + (z'-f)^2 d^2}\right)\right] \int I_\lambda \, dz \qquad (9)$$

Equations (8) and (9) may both be further simplified by noting that $d << \bar{z}$ or $d << z'$ and that, for small values of the argument, the arcsin is approximately equal to the argument. With these substitutions, Eqs. (8) and (9) become

$$E_\lambda = \frac{w h d^2}{4\pi \bar{z}^2} \int I_\lambda \, dz = \frac{w' h' d^2}{4\pi z'^2} \int I_\lambda \, dz \qquad (10)$$

This equation can be stated in either of two ways. First, the light energy reaching the grating of a spectrometer is proportional to the area of the slit image, the solid angle of the lens, or collimator image when viewed from the slit image, and the integral of the variation of the intensity of the emitted light along the axis of the spectrometer; second, the light energy which reaches the grating of the spectrometer is proportional to the area of the slit, the solid angle of the lens or collimator when viewed from the slit, and the integral of the variation of the intensity of the emitted light along the axis of the spectrometer.

SECTION III
DISCUSSION

Because the integrand of the final integral to be evaluated is only the intensity function, the axial variations in emitted intensity cannot be detected by an axial shift of the optical system. For a circularly symmetrical emitting field, an Abel inversion technique[1] can be used. For non-circular fields, a similar technqiue in which sets of observations are made with the optical axis aligned in two perpendicular directions will permit analysis. The accuracy of this analysis will be somewhat less than that of the Abel inversion but can be improved to a satisfactory level by an increase in the number of observations.

Equation (10) shows that, while the field of view of a spectrometer has the width and height of the slit image at the slit image and increases

[1]Griem, Hans R. Plasma Spectroscopy. New York, McGraw-Hill, 1964, p. 176.

in area in both directions along the z axis approximately as the square of the distance from the slit image, the fraction of the emitted energy reaching the grating is inversely proportional to the square of the distance, and the resultant energy reaching the grating is therefore independent of the distance.
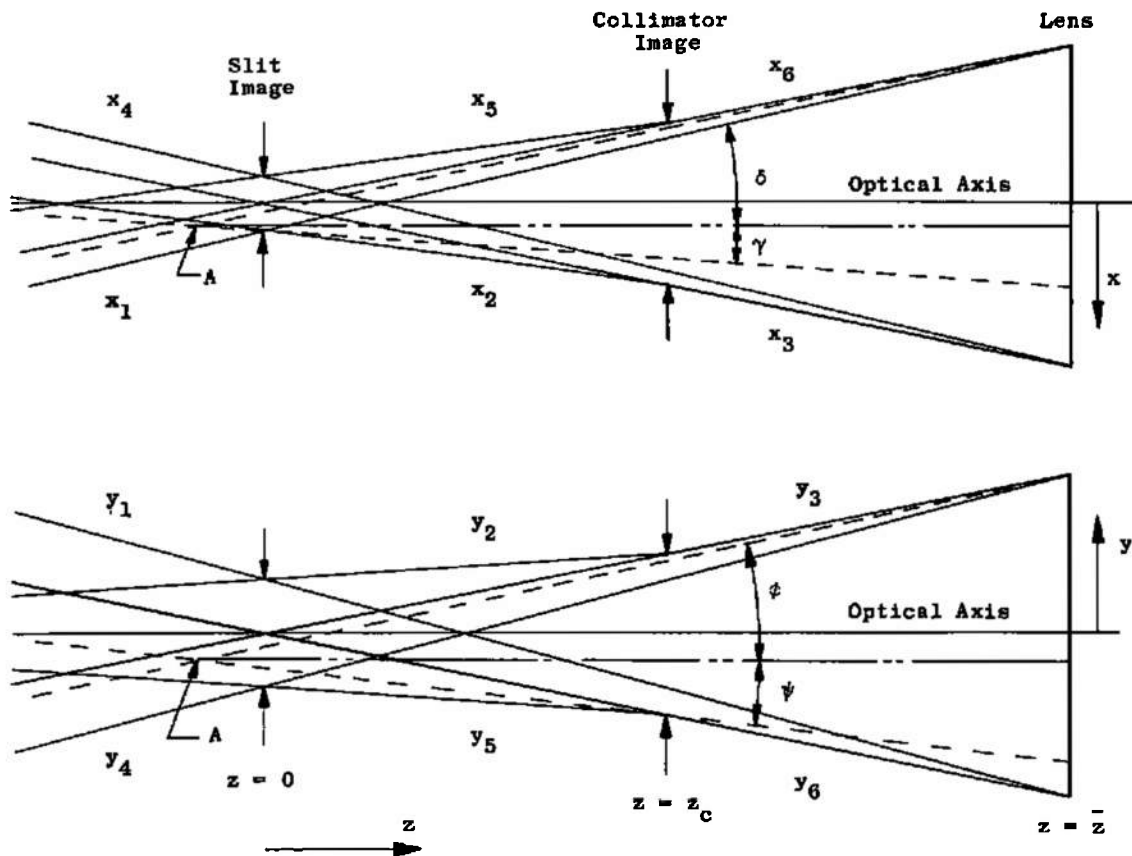
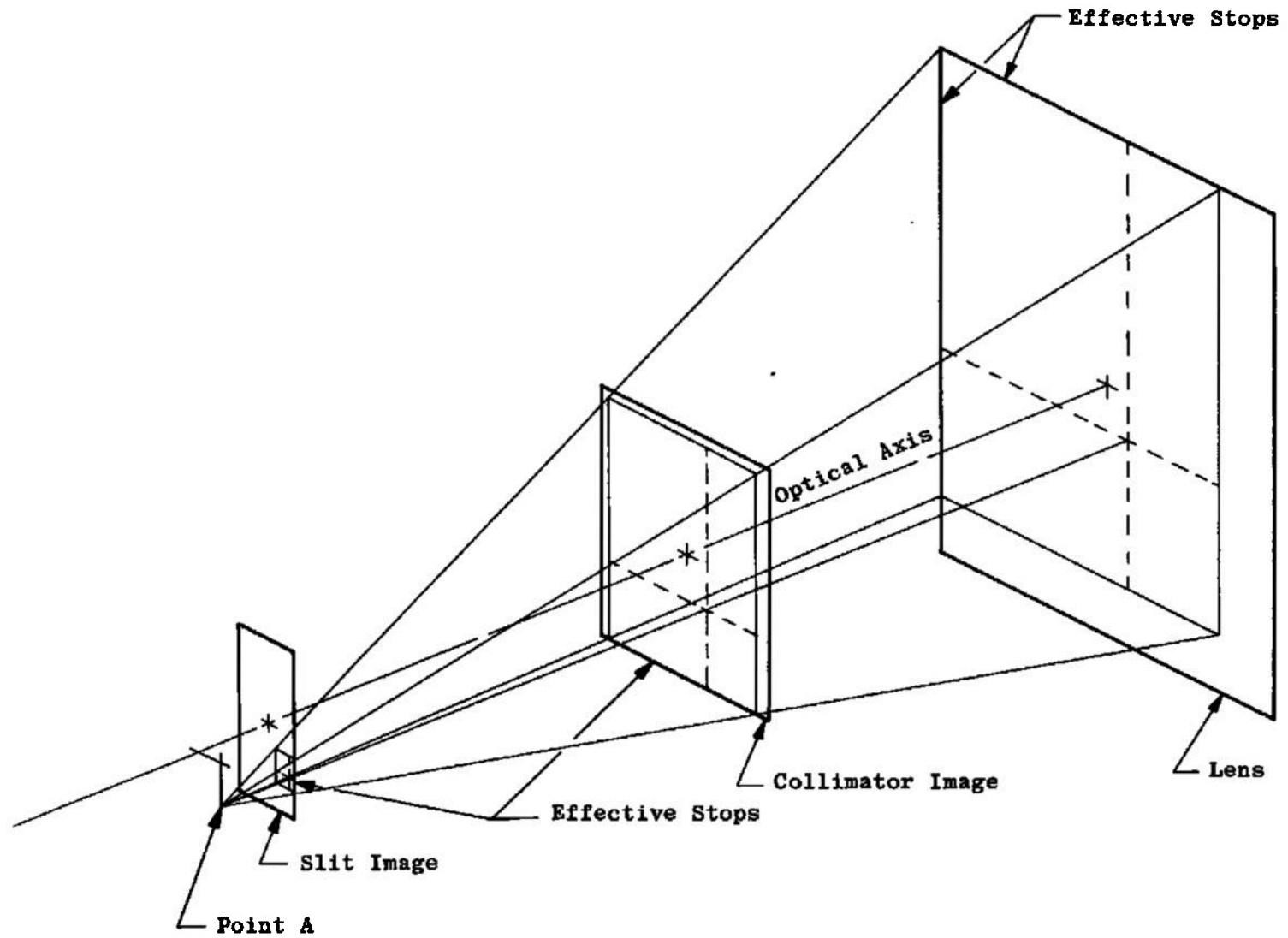Fig. 1 Optical Diagram Showing Limiting Planes for the Determination of the Effective Stops

Effective Stops

Optical Axis

Collimator Image

Effective Stops

Lens

Slit Image

Point A

Fig. 2 Optical Projection of the Effective Solid Angle for a Point
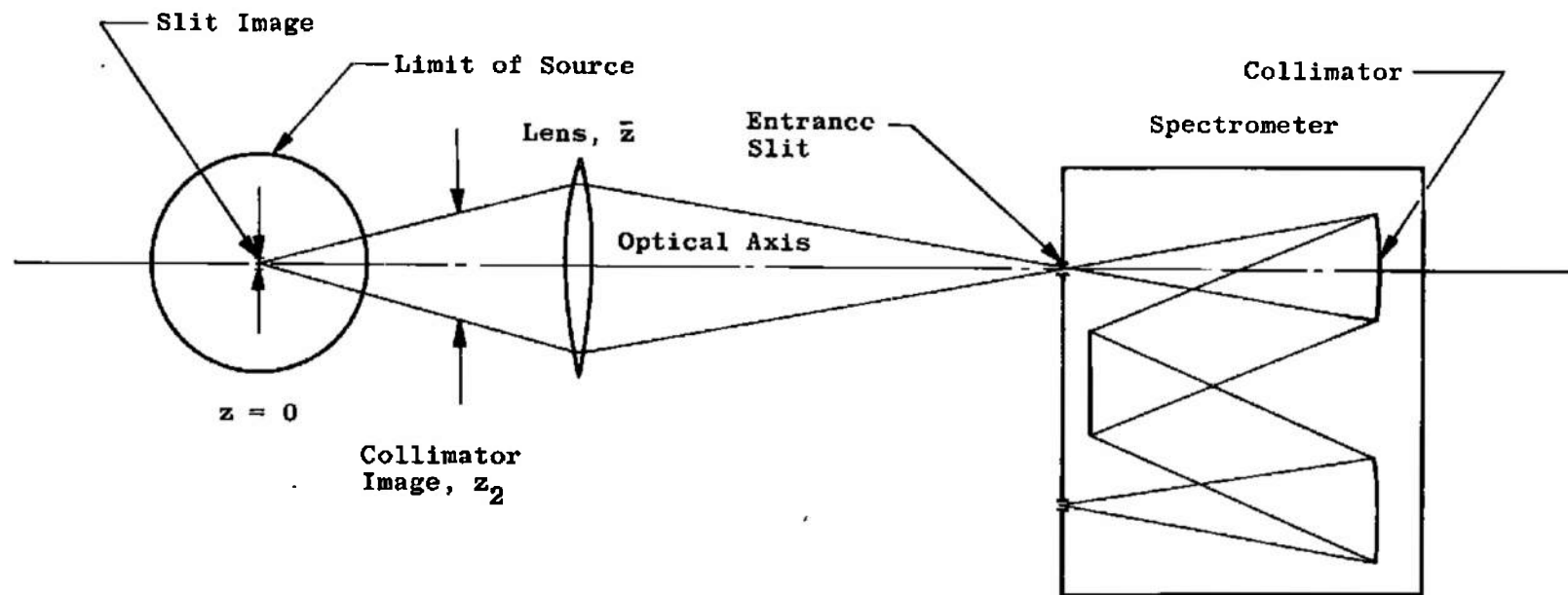
Fig. 3 Typical Layout of Apparatus

TABLE I
LIMITING PLANES

Horizontal Limits

$$x_1 = \frac{w}{2} - \frac{w+d}{2} \cdot \frac{z}{z}$$

$$x_2 = \frac{w}{2} + \frac{d_c - w}{2} \cdot \frac{z}{z_c}$$

$$x_3 = \frac{d_c}{2} \cdot \frac{z}{z_c}$$

$$x_4 = -\frac{w}{2} + \frac{w+d}{2} \cdot \frac{z}{z}$$

$$x_5 = -\frac{w}{2} - \frac{d_c - w}{2} \cdot \frac{z}{z_c}$$

$$z_6 = -\frac{d_c}{2} \cdot \frac{z}{z_c}$$

Vertical Limits

$$y_1 = \frac{h}{2} - \frac{h+d}{2} \cdot \frac{z}{z}$$

$$y_2 = \frac{h}{2} + \frac{d_c - w}{2} \cdot \frac{z}{z_c}$$

$$y_3 = \frac{d_c}{2} \cdot \frac{z}{z_c}$$

$$y_4 = -\frac{h}{2} + \frac{h+d}{2} \cdot \frac{z}{z}$$

$$y_5 = -\frac{h}{2} - \frac{d_c - h}{2} \cdot \frac{z}{z_c}$$

$$y_6 = -\frac{d_c}{2} \cdot \frac{z}{z_c}$$

$$\frac{d_c}{z_c} = \frac{d}{z}$$

TABLE II
LIMITING ANGLES

z Range	x Range	Horizontal Angle
$-\infty < z < 0$	$x_2 < x < x_1$	$\gamma_{21} = \tan^{-1}\left(x - \frac{w}{2}\right)/z$
$-\infty < z < z_c$	$x_3 < x < x_2$	$\gamma_{32} = \tan^{-1}\left(\frac{d_c}{2} - x\right)/(z_c - z)$
$-\infty < z < z_c$	$x_4 < x < x_3$	$\gamma_{43} = \tan^{-1}\left(\frac{d}{2} - x\right)/(\bar{z} - z)$
$0 < z < z_c$	$x_5 < x < x_4$	$\gamma_{54} = \tan^{-1}\left(\frac{w}{2} + x\right)/z$

z Range	y Range	Vertical Angle
$-\infty < z < 0$	$y_2 < y < x_1$	$\phi_{21} = \tan^{-1}\left(y - \frac{h}{2}\right)/z$
$-\infty < z < z_c$	$y_3 < y < x_2$	$\phi_{32} = \tan^{-1}\left(\frac{d_c}{2} - y\right)/(z_c - z)$
$-\infty < z < z_c$	$y_4 < y < x_3$	$\phi_{43} = \tan^{-1}\left(\frac{d}{2} - y\right)/(\bar{z} - z)$
$0 < z < z_c$	$y_5 < y < x_4$	$\phi_{54} = \tan^{-1}\left(\frac{h}{2} + y\right)/z$

TABLE III
LIMITS OF INTEGRATION AND z FUNCTIONS

a. z < 0

β – Limits

	$-\dfrac{d+h}{2\bar{z}}$	$\dfrac{d_c-h}{2z_c}$	$\dfrac{d}{2\bar{z}}$	$\dfrac{d+h}{2\bar{z}}$
α – Limits				
$-(d+w)/2z$				
	$(\ -z)(\ -z)$	$(z_c-z)(\ -z)$	$(\bar{z}-z)(\ -z)$	
$(d_c-w)/2z_c$				
	$(\ -z)(z_c-z)$	$(z_c-z)(z_c-z)$	$(\bar{z}-z)(z_c-z)$	
$d/2\bar{z}$				
	$(\ -z)(\bar{z}-z)$	$(z_c-z)(\bar{z}-z)$	$(\bar{z}-z)(\bar{z}-z)$	
$(d+w)/2\bar{z}$				

b. 0 < z < z_c

β – Limits

	$=\dfrac{d_c-h}{2z_c}$	$\dfrac{d_c-h}{2z_c}$	$\dfrac{d}{2\bar{z}}$	$\dfrac{d+h}{2\bar{z}}$
α – Limits				
$-(d_c-w)/2z_c$				
	$z \cdot z$	$z_c \cdot z$	$\bar{z} \cdot z$	
$(d_c-w)/2z_c$				
	$z \cdot z_c$	$z_c \cdot z_c$	$\bar{z} \cdot z_c$	
$d/2\bar{z}$				
	$z \cdot \bar{z}$	$z_c \cdot \bar{z}$	$\bar{z} \cdot \bar{z}$	
$(d+w)/2\bar{z}$				

TABLE IV
MODIFIED z FUNCTIONS AND LIMITS OF INTEGRATION

a. z < 0

β – Limits

	$\dfrac{d_c - h}{2\,z_c}$	$\dfrac{d}{2\,\bar{z}}$	$\dfrac{d + h}{2\,\bar{z}}$
α – Limits			
$(d + w)/2\,\bar{z}$			
	$z_c \cdot \bar{z}$	$z \cdot \bar{z}$	
$d/2\,\bar{z}$			
	$z_c \cdot z_c$	$\bar{z} \cdot z_c$	
$(d_c - w)/2\,z_c$			

b. 0 < z < z_c

β – Limits

	$\dfrac{d_c - h}{2\,z_c}$	$\dfrac{d}{2\,\bar{z}}$	$\dfrac{d + h}{2\,\bar{z}}$
α – Limits			
$(d + w)/2\,\bar{z}$			
	$z_c \cdot \bar{z}$	$\bar{z} \cdot \bar{z}$	
$d/2\,\bar{z}$			
	$z_c \cdot z_c$	$\bar{z} \cdot z_c$	
$(d_c - w)/2\,z_c$			

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13. ABSTRACT

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where

w = width of slit image

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d = width of lens stop

$\bar{z}$ = distance from slit image to lens

I_λ = intensity of light emitted

z = coordinate distance from slit image toward lens

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14. KEY WORDS	LINK A		LINK B		LINK C	
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spectrometers						
lens						
wavelength						
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